1. My Pattern is Better than Yours

a) Flip a fair coin twice. What is the probability that you get two heads (HH)? What is the probability that you get heads followed by tails (HT)? Are these probabilities the same?
b) Flip a fair coin repeatedly until you get two heads in a row (HH). On average, how many flips should this take? What if we flip until we get heads followed by tails (HT)? Are the answers the same?
c) Let's play a game: we flip a coin repeatedly until either HH emerges (I win) or HT emerges (you win). Is the game fair?
d) We play the let's-flip-a-coin-until-a-pattern-emerges game. You pick HHT as your pattern, I pick THH. We flip a fair coin repeatedly until we get heads-heads-tails in a row (you win) or tails-heads-heads in a row (I win). Is the game fair?
e) Seeing as THH is a better pattern, you request to pick it as your pattern. I graciously agree, and switch to TTH. I keep beating you most of the time. You switch to my TTH. I switch to HTT. I keep beating you. You switch to HTT. I switch to... HHT, your original losing pattern. Who's winning now?
This delightful result is known as Penney's game.

Answer: https://www.quora.com/What-are-the-solutions-to-Penneys-game-and-what-is-the-explanation

2. Tuesday Girl

Many people are familiar with the old chestnut: pick a random family uniformly among all families with exactly 2 children of which one (at least) is a girl. What is the likelihood that the chosen family has 2 girls? Under the usual assumption of gender uniformity at birth, the answer is 1/3, not 1/2 as many people initially assume. This is fairly well known, but is still a useful warm-up.

Let's up the ante: what if we pick a random family among all families with exactly 2 children, at least one of which is a *girl born on a Tuesday*?

It's best to explicitly state that twins are to be ignored here.
This fun problem made the rounds circa 2010, apparently starting with a talk by Gary Foshee at a Gathering for Gardner meeting. When intuition and math probably look wrong is one of many write-ups.

Answer:

The solution space is coloured green.

GTue GMon	GTue GTue	GTue GWed	GTue GThu	GTue GFri	GTue GSat	GTue GSun
GWed GMon	GWed GTue	GWed GWed	GWed GThu	GWed GFri	GWed GSat	GWed GSun
GThu GMon	GThu GTue	GThu GWed	GThu GThu	GThu GFri	GThu GSat	GThu GSun
GFri GMon	GFri GTue	GFri GWed	GFri GThu	GFri GFri	GFri GSat	GFri GSun
GSat GMon	GSat GTue	GSat GWed	GSat GThu	GSat GFri	GSat GSat	GSat GSun
GSun GMon	GSun GTue	GSun GWed	GSun GThu	GSun GFri	GSun GSat	GSun GSun
GMon BMon	GMon BTue	GMon BWed	GMon BThu	GMon BFri	GMon BSat	GMon BSun
GTue BMon	GTue BTue	GTue BWed	GTue BThu	GTue BFri	GTue BSat	GTue BSun
GWed BMon	GWed BTue	GWed BWed	GWed BThu	GWed BFri	GWed BSat	GWed BSun
GThu BMon	GThu BTue	GThu BWed	GThu BThu	GThu BFri	GThu BSat	GThu BSun
GFri BMon	GFri BTue	GFri BWed	GFri BThu	GFri BFri	GFri BSat	GFri BSun
GSat BMon	GSat BTue	GSat BWed	GSat BThu	GSat BFri	GSat BSat	GSat BSun
GSun BMon	GSun BTue	GSun BWed	GSun BThu	GSun BFri	GSun BSat	GSun BSun
BMon GMon	BMon GTue	BMon GWed	BMon GThu	BMon GFri	BMon GSat	BMon GSun
BTue GMon	BTue GTue	BTue GWed	BTue GThu	BTue GFri	BTue GSat	BTue GSun
BWed GMon	BWed GTue	BWed GWed	BWed GThu	BWed GFri	BWed GSat	BWed GSun
BThu GMon	BThu GTue	BThu GWed	BThu GThu	BThu GFri	BThu GSat	BThu GSun
BFri GMon	BFri GTue	BFri GWed	BFri GThu	BFri GFri	BFri GSat	BFri GSun
BSat GMon	BSat GTue	BSat GWed	BSat GThu	BSat GFri	BSat GSat	BSat GSun
BSun GMon	BSun GTue	BSun GWed	BSun GThu	BSun GFri	BSun GSat	BSun GSun

There are 27 possibilities (that include a Girl born on a Tuesday.) Of these, almost half (i.e. 13) are Girl-Girl.

From 1 in 3 (33.3%) to 13 in 27 (48.1%) just by positing a Tuesday birthday.

3. The Next Card

I shuffle a deck of cards and deal them one by one, as slowly as you need me to. You observe the sequence of cards and at any point of your choosing you say Stop. I then deal the next card: if it's black, you win. If it's red, you lose. No jokers and no sleight-of-hand. If you fail to say Stop until the very end, the last card determines the outcome of the game.

What's your strategy?

Answer: No Strategy will work. In other words, all strategy will give same result. The rules are: the cards, shuffled uniformly, are dealt one by one. You can say "stop" at any moment, and you win if and only if the next card is black. You don't get to guess the color of the next card and then win if you're right. If that were the case, the game would certainly have good strategies.

Also, if you don't say "stop", the last card determines whether or not you win.

With that out of the way, here's a very simple, intuitive explanation for why no strategy makes any difference in this game.

1) Suppose the game was slightly different: the cards are dealt, you say stop, and then we check the last card in the deck. If it's black, you win.

Does "strategy" matter in this modified game? Of course not. This game is plainly ridiculous. The dealing and the stopping do nothing to the color of the last card. We shuffle the deck, and whatever the last card is directly determines whether or not you win or lose. It's just like flipping a fair coin.

2) The modified game and the original game are the same game. When you say "stop", the chances of the next card being red are the same as the chances of the last card being red. In fact, all face-down cards are completely symmetric, and their location in the deck has no relationship

with their color (This is the most important part, that the location of the card and the color of the card are two events that are independent of each other).

4. The Disease Test

So there's this disease named X that 1% of the population has.
Disease X shows no symptoms.
Thus, a special medical test has been created to determine whether a subject has it.
The test has a 98% accuracy for positive results and 97% accuracy for negative results.
For example, if a person is sick then there's 98% chance the test will come out as positive and 2% as negative.
Similarly, if a person is healthy then there's 97% chance the test will come out as negative and 3% as positive.

The question is: You took the test and it came back positive. What are the chances that you are sick?

Answer: The given probability of 98% is the chance of the test returning positive given that the person is sick. The question asks what is the chance of being sick given that the test returned positive.

Let's list the cases of getting a positive result:
[1] People who are sick (1%) and the tests were accurate (98%).
[2] People who are healthy (99%) but the tests were inaccurate(3%).

So according to [1] almost 1% of the population will get a true positive. (0.01 * 0.98 to be exact)
And according to [2] almost 3% of the population will get a false positive. (0.99 * 0.03 to be exact)
Meaning, that within every 4 people that receive a positive result, only one of them is actually sick (or 25%).

This can also be verified using Bayes Theorem of conditional probability.

5. Comparing the numbers

Let's play a game. I pick a probability distribution on the real line. You know nothing about it, except that it's everywhere nonzero.

Using my probability distribution, I generate two distinct numbers. Then I flip a fair coin to determine which number to show you -- call it A, and call the other number B. Then I ask you whether A < B or B < A?

Certainly, you can just guess. And you'd have a 50 percent chance of winning.

But is there a strategy to do better?

Answer:

Consider this strategy: if the number I show you is less than 0, you guess it's the lesser of the two numbers. If it's greater than 0, you guess that it's the greater.

But there's no reason to assume the distribution has a mean of 0! There's no reason to assume it has a mean at all!
Right. Your number doesn't have to be 0. I could just as easily have said 37 or $\sqrt{\pi}$. But it's just some arbitrary line in the sand.

What's your probability of success?

Call the numbers I generate A and B with $A<B$. There are a few possibilities about their relationship with 0:

Case 1: $0<A<B$.
In this case, you win with 50% probability. Both numbers are greater than 0, so you'll always guess that I'm showing you the greater of the two. Whether you're right or wrong depends on the flip of a fair coin: if I show you B, you win. If I show you A, you lose.

Case 2: $A<B<0$.
Same deal as above: you always guess I'm showing you the lesser number. You're right with 50% probability.

Case 3: $A<0<B$
You win with 100% probability. If I show you A, you (correctly) say it's the lesser number. If I show you B, you (correctly) say it's the greater number!
Moreover, this case occurs **at least sometimes**, because I told you that the probability distribution is everywhere nonzero!

6. The Gift of the Dinner Party

Suppose you are at a dinner party. The host wants to give out a door prize that is wrapped in a box. Everyone (including the host) sits around a circular table and each person is given a fair coin. Initially the host is holding the box. He flips his coin. If it is heads, he passes to the right. If it is tails, he passes to the left. The process is repeated by whichever guest is holding the box. (Heads, they pass right; tails, they pass left.) The game ends when the last person to receive the box finally gets it for the first time. That person gets to keep the box as the winner of the game.

Where should you sit to maximize your chance of winning?

Answer: The answer to this problem is you can sit anywhere! Your chances are equally likely to win if you're sitting next to the host or across the table from the host, the game would just take longer if the winner was on the other side of the table because the box would have to go around at least one and a half times.

7. Did You Get Your Seat?

Suppose you have a 100-seater airplane, and each of 100 passengers has a boarding pass with an assigned seat, and they enter the airplane one by one in a random order. The first passenger is a jerk, however, and just sits down wherever he wants, without regard to the seat number on his ticket. All the other passengers, being exceedingly polite, either take the seat assigned to them if available, but if someone is already sitting there, take a random empty seat rather than bother the squatter.

What is the probability that the last passenger to board will be able to sit in his assigned seat?

Answer: The probability, somewhat surprisingly, is 1/2! There are long ways to see this, and short ways ... let me describe what I think is the shortest solution that is still fairly complete.

The key observation is this: when the last person boards, the only possibilities for empty seats are the correct seat, or the seat assigned to the first person. Why? well, if the seat assigned to the 16th person to board is free when the last person boards, then it was also free when the 16th person boarded, so she would have taken it then, a contradiction; and the same contradiction works for everyone else after the first person to board.

A corollary of this observation is that whenever a passenger makes a random choice, both the first person's and last person's seat must be available. For if, after one of these seats is taken, a passenger comes on and finds that she has to make a random selection between many seats, there is a non-zero probability that she takes the other of these two special seats, contradicting the key observation (since it forces the last passenger to sit somewhere other than her own seat or the first person's seat, a situation we now know to be untenable).

Armed with the key observation, we see that the event that the last person's correct seat is free, is exactly the same as the event that the first person's seat was taken before the last person's seat. What could be the probability of that? Well, each person who came on the plane and had to make a random choice, was equally likely to choose the first person's seat or the last person's seat - the random chooser exhibits absolutely no preference towards a particular seat. This means that the probability that one seat is taken before the other must be 1/2.

Another way to think of this last part: if you tried to write down two exact expressions, one (A) for the probability that the first person's seat is taken before the last person's seat, and one (B) for the probability that the last person's seat is taken before the first person's seat, these two expressions would have to be identical, since every time a random choice is made, the probability of the first person's seat being chosen is equal to the probability of the last person's seat being chosen. Since A=B, and this covers all possibilities (by the key observation), they must both be equal to 1/2.

I learned this puzzle from Peter Winkler's book ``Mathematical Puzzles (A Connoisseur's Collection)'' (where he gives the solution described above). It also appears in Bela Bollobas' ``The Art of Mathematics (Coffee Time in Memphis)'', where a different solution is given.